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Pragmatic Insights into Set-Theoretic Independence

Exploring Disagreement and Agreement among Practitioners

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Berlin, December 2024 *Deborah Kant*

Preface

This book started as a project on naturalness in set theory. Early in my PhD, I focused on the idea that perhaps set-theoretic practitioners, whether they favour a universe or multiverse view, agree on their naturalness judgements. Thus, it seemed that one simply had to extract these naturalness judgements, identify the new axioms they apply to, and propose these as new axioms for ZFC. However, the initial progress revealed a significant problem: defining naturalness proved to be extremely difficult, as did accessing the naturalness judgements of set-theoretic practitioners.

During this period, I explored other fascinating topics related to naturalness. In the literature, a natural sentence is often understood as a mathematical one, as opposed to a meta-mathematical sentence such as Gödel sentences. I delved into the formalities of separating the object level from the meta level in set theory, aiming to apply this separation to every step of a forcing proof to assess the naturalness of forcing.

This retrospective on the early developments of my thesis illustrates that the project allowed for varied methodological approaches: an analytic explication of 'e is natural' guided by prototypical examples and my own intuitions, a formal treatment of separating the object level from the meta level, an informal specification of naturalness based on conversations with set theorists, and an analysis of naturalness judgements in set-theoretic discourse based on data collected from rigorous interviews with practitioners.

I experienced several shifts in thinking, ultimately opting for the interview study as I became convinced that the problem I wanted to address required this method. To study the naturalness judgements of set theorists, I needed to collect corresponding data. Using other methods, I would not have been able to claim that my analysis was in accordance with set-theoretic practice.

Once I made this decision, conducting the interviews was great fun. It was fascinating and interesting to talk to professional set theorists about their work and their views on their work. I could ask all the questions I was interested in, and my interview partners were open to sharing their thoughts and explaining their perspectives. Step by step, an overview of the variety of set-theoretic practices emerged, with the philosophically most relevant parts now documented in this book.

This final version of my PhD thesis differs in several ways from the version I submitted to the University of Konstanz in June 2022. I restructured the chapters, included more background information, and revised all parts in detail to make the content more accessible to interested readers.

Introduction

This book addresses a key philosophical problem in the philosophy of mathematics: a problem about mathematical truth. Most people think that mathematics has a clear-cut notion of truth: mathematical statements are either true or false, and there is no room for debate. Other fields of knowledge, even the sciences, do not have such a clear notion of truth. However, the *set-theoretic independence phenomenon* shakes up the idea that mathematical truth is without problems.

Set theory has served as the foundation of mathematics for more than a hundred years. The concepts, language, and axioms of set theory have turned out to be mathematically fruitful in various foundational senses: every mathematical sentence can be expressed in set-theoretic language; arbitrary mathematical objects can be interpreted as sets; and mathematical questions can be solved by proof or refutation from the set-theoretic axioms.

An important and distinguishable asset of set theory is its ability to handle rigorously infinite sets. Fundamental number structures, including the natural numbers, integers, the rational numbers, and real numbers, can be precisely defined using set theory. But by delving deeper into the world of infinite sets, mathematicians and logicians found that set theory can explain only a few facts about infinite sets. Other questions seem to be out of its range: the famous continuum hypothesis, projective determinacy, the existence of large cardinals, Suslin's hypothesis, and many other statements about infinite sets are neither true nor false according to the standard axiomatic theory of sets – the Zermelo-Fraenkel-axiomatisation with the axiom of choice (ZFC).

This inability of ZFC to definitively resolve many questions about infinite sets is a mathematically interesting phenomenon. To prove that a given statement is consistent with ZFC, set theorists build a model in which the statement is true. But what if they can also build a model in which the statement is false? Then the statement is neither true nor false: it is *independent*. Mathematical truth in set theory is not clear-cut at all. There is considerable room for debate.

Many philosophers, mathematicians, and logicians have stated their diagnoses of this situation and presented their philosophical views on how to

understand it, and sometimes how to solve it. One prominent voice is that of Kurt Gödel (1906–1978),[1] who argued that the now-standard theory ZFC should be extended by further axioms. However, these new axioms cannot be arbitrarily chosen: they should be justified, either intrinsically by relating them to the concept of set, or extrinsically by external factors, such as the convincing and desirable consequences of the axiom. This proposal is called Gödel's programme, and several philosophers and set theorists have taken it up and developed it further.

The set-theoretic researcher W. Hugh Woodin argues that there is a unique set-theoretic universe V in which every sentence is either true or false, and set theory is the endeavour to find out which ones are true and which ones are false. His large research programmes are designed to realise these goals. On the other hand, the set-theoretic researcher Joel D. Hamkins argues that this view might have been plausible before the set-theoretic independence phenomenon was discovered, but given the numerous set-theoretic models that are investigated by practitioners today, we must assume instead that there is an enormous set-theoretic multiverse that encompasses all set-theoretic models, and set theory is the endeavour to explore this rich multiverse and find out what is true in it. As these contrasting positions show, the set-theoretic independence problem is not easily resolved.

Although intrinsic and extrinsic justifications are still widely explored, one shortcoming is that they are not particularly representative of set-theoretic research experience. For a set-theoretic practitioner, axioms can be plausible, obvious, useful, or natural, but whether an axiom is justified on intrinsic or extrinsic grounds is sometimes hard to say. To overcome this shortcoming, I study the idea of *naturalness*. Set-theoretic practitioners can usually say whether they find an axiom natural or not. At first glance, all set theorists seem to have opinions on naturalness, whether they believe in V or not. The proposal is to reduce the problematic independence phenomenon by isolating from the wide range of possible new axiom candidates those that set-theoretic practitioners find natural and adding them to ZFC. My main goal in this book is to evaluate whether this naturalness attempt towards the set-theoretic independence problem can work.

First, we need to clarify the starting positions from which set-theoretic practitioners consider the set-theoretic independence phenomenon. A basic question is: do set-theoretic practitioners have philosophical views about the independence phenomenon? Because Woodin and Hamkins, both practising set theorists, disagree fundamentally on their views, a second important question is: do set-theoretic practitioners disagree on their philosophical views about the independence phenomenon? Disagreement would probably make it considerably more difficult to solve the problem. Answering these questions clears the initial ground.

[1] I provide the dates of birth and, if applicable, death for individuals born before 1950.

Two other issues must also be investigated because they identify necessary requirements for the naturalness attempt to work. Naturalness judgements on their own are not sufficient to solve the set-theoretic independence problem: they must be related to a change of attitude. Consequently, the third question is: if set-theoretic practitioners find an axiom natural, do they also find it acceptable? Finally, sufficient agreement on naturalness judgements is necessary. If set-theoretic practitioners all disagreed on their naturalness judgements, no new axiom candidate could be isolated. Therefore, the fourth question is: do set-theoretic practitioners agree on their naturalness judgements about axioms? These four main questions will lead us through the book.

In Chapter 1, I shall first explain the set-theoretic independence phenomenon. Then, I shall draw out the set-theoretic independence problem and present different proposals on how to understand or solve it. I also explain the naturalness attempt in more detail and review what is known about naturalness judgements in mathematics. For instance, the literature describes naturalness as a prescriptive notion with a positive connotation (San Mauro and Venturi, 2015). This supports the view that the naturalness attempt could work: a natural axiom would be a good axiom that one should adopt.

Once the research questions are fixed, one must choose an adequate method to address them. Several conceptual analyses and ontological and epistemological proposals have been made in the debate on the set-theoretic independence problem, but among these different approaches there is no substantial pragmatic approach. This is problematic because a purely theoretical debate does not sufficiently acknowledge the fact that it is the set theorists themselves who will decide on new axiom candidates. The community of philosophers of mathematics cannot make this decision.

This book endorses exactly this fact. The set-theoretic community is decisive for solving the set-theoretic independence problem, and a pragmatic approach can best take into account the significance of the set-theoretic community. The idea of pragmatism is to evaluate philosophical claims against empirical facts. If only a few set-theoretic practitioners believed in V, then, from a pragmatic perspective, this would be evidence against the claim that there is a unique set-theoretic universe V. If set-theoretic practitioners do not accept the axioms that they find natural, then, from a pragmatic perspective, the naturalness attempt does not work.

Fortunately, this research project can build on important work in the philosophy of mathematical practice and in social epistemology. For around 50 years, philosophers have increasingly emphasised the significance of topics relevant to mathematical practitioners. For instance, a mathematical proof was, for a long time, explained as a formal derivation in a formal system. However, proofs of this type could not be found in mathematics journals. Mathematical proofs are formulated in natural language using some for-

mal technicalities; they are thus far removed from a formal derivation in a formal system. Philosophers of mathematical practice emphasise and attempt to resolve tensions between theoretical concepts and their empirical counterparts by proposing more adequate explanations.

Social epistemology investigates the inner workings of groups that strive for knowledge. Its fundamental position is that knowledge-seeking is a social endeavour and that philosophers should understand and explain the mechanisms guiding this social endeavour. This approach is applicable to scientific communities of the modern age, in which scientists usually collaborate, sometimes in large research groups. Social epistemologists have already thought about disagreement and agreement in scientific communities. An especially problematic phenomenon is *deep disagreement* between scientists. Given that deep disagreements are about what constitutes a reason, they are not easily resolved.

In the philosophy of set theory, a few attempts towards considering set-theoretic practice have been made. Most prominently, Penelope Maddy has investigated set-theoretic practice for years, argued for its philosophical significance, and developed a comprehensive philosophical theory about the ontology and epistemology of set theory. According to her analysis, the universe view, also endorsed by Woodin, is correct. Set theorists aim to extend ZFC by new axioms, and they justify these axioms on extrinsic grounds. In her seminal articles and books, Maddy examines in detail specific reasons in favour of or against axiom candidates suggested by set theorists, and she explains and defends her meta-philosophical approach that underlines the significance of set-theoretic practice, as well as provides a theory of set-theoretic epistemology and ontology. Hamkins also considers set-theoretic practice. He explains that set-theoretic practitioners today explore a rich world of set-theoretic models. He is himself a set-theoretic practitioner and therefore provides an inner view of the set-theoretic community. However, his conclusions differ greatly from those of Maddy. Colin Rittberg noticed this discrepancy and argued that Maddy considers not the whole set-theoretic practice, but only parts of it. Rittberg examined in detail the different set-theoretic practices of Woodin and Hamkins.

The contrasting philosophical conclusions of these attempts are one problem. Another problem, which might explain the first, is that their analyses of set-theoretic practice are mostly based on only a few perspectives. Maddy focused mostly on members of the Cabal, a group of set theorists very active in California who do indeed search for new axioms. Hamkins certainly has knowledge of the set-theoretic community but mainly provides his own diagnosis of the situation. Rittberg acknowledges that there seem to be quite different perspectives in set-theoretic practice but presents specific case studies of only a few set-theoretic practitioners. What might resolve these problems is an overview of the set-theoretic community that investigates the research practices of set-theoretic practitioners from a vari-

ety of different backgrounds. Therefore, the method chosen to address the research questions is an exploratory interview study with 28 set-theoretic practitioners from different research backgrounds.

In Chapter 2, I explain the pragmatic approach, review relevant literature in the philosophy of mathematical practice and social epistemology, and compare the approach pursued in this book with Maddy's approach. This concludes part I.

Part II presents the interview study; in Chapter 3 the methodology, and in Chapter 4 the results. In the interview study, I asked 28 set-theoretic practitioners about their research area, the use of new axioms in their work, forcing, the possibility of extending ZFC by new axioms, and naturalness judgements in set-theoretic practice. The questionnaire was designed to stay close to the usual set-theoretic research discourse; it avoids questions that are too philosophical. The sociological methodology used is built on Philipp Mayring's qualitative content analysis, a systematic and transparent method of analysing qualitative data. I adapted Mayring's rules to the specifics of the set-theoretic context and my research questions. The resulting methodology is explained in Chapter 3, and further information on my method is to be found in the Appendices.

Chapter 4 presents the results of the interview study that are relevant to the four main research questions. It includes a detailed presentation of the philosophical views of the participants and all information on naturalness judgements. Data on the spectrum of disagreement and agreement in both areas are included. Chapter 4 also presents results about surprising theorems. This is an unplanned topic relevant to the research questions that surfaced during the pilot study. This chapter covers only some of the results of the study; a full treatment of all the qualitative data collected in the study lies outside the scope of this book. Other results of the study are analysed in further publications such as (Kant, 2025).

For readers who are mainly interested in the philosophical outcome of the interview study, Chapters 3 and 4 can be skipped and consulted when useful, although I recommend scanning the description of the sample set in 3.5. The results are summarised in Part III.

Part III, the main part of the book, discusses the results of the study and presents philosophical conclusions and hypotheses. Methodologically, the qualitative data are strong in *suggesting* new connections, phenomena, and relevant factors. However, they are weak in *proving* these hypotheses. To prove or refute a conjecture in the social sciences, one must conduct a representative study. This qualitative study provides hypotheses for such a study. This book, therefore, proposes a theory that describes connections, phenomena, and relevant factors in the set-theoretic community regarding the set-theoretic independence problem and naturalness judgements. This theory is strongly suggested by the results of the qualitative interview study.

Chapter 5 tackles the philosophical views of set-theoretic practitioners,

relates them to the specific research areas, and discusses disagreement and agreement. This chapter answers the questions of whether set-theoretic practitioners have philosophical views about the independence phenomenon and whether they disagree on these views. The question of whether deep disagreement is found within the set-theoretic community is also discussed. This provides a well-informed overview of the philosophical beliefs of set-theoretic practitioners.

Chapter 6 proposes a detailed account of the role of naturalness judgements in set-theoretic practice. It brings together the scattered pieces of information included in the study to form a bigger, more coherent picture. This picture highlights the social aspects of naturalness judgements, shows how naturalness relates to familiarity, simplicity, and truth, and addresses the spectrum of disagreement and agreement. The proposed account embeds naturalness judgements in an ongoing research process and explains the development, change, and significance of naturalness judgements.

Following on from this general picture of the role of naturalness judgements, Chapter 7 examines in detail naturalness judgements about axioms. A detailed case study investigates naturalness judgements about forcing axioms. This chapter answers the research questions of whether set-theoretic practitioners find an axiom acceptable if they find it natural, and whether they agree on their naturalness judgements about axioms. Part III concludes with an overall evaluation of whether the naturalness attempt to solve the set-theoretic independence problem works from a pragmatic perspective.

The last part of the book offers two additional titbits. Chapter 8 explores the suggested tendency of my conclusions and goes beyond them by proposing a more general account. It proposes a differentiation of different discourse layers in the set-theoretic research discourse, separating the discourses about value judgements from those about mathematical and philosophical beliefs. This view also illuminates the issue of extrinsic justification.

Finally, Chapter 9 contributes an analysis of a real-world case study for the social epistemological debate on *peer disagreement*. In this debate, philosophers ask whether two scientists with similar expertise (epistemic peers) who disagree on some proposition are required to change their beliefs because the disagreement with an epistemic peer provides evidence to do so. We can consider some set-theoretic practitioners to be involved in such a peer disagreement.

Having outlined the book to come, I am looking forward to leading you through it. If you are interested only in some parts, feel free to jump in at any point. I shall lay out in detail my thought process and I invite you to take with you whatever is useful for your thought process.